

Week 12 Worksheet - Convergence Test of Series

Instructions. Follow the instructions of your TA and do the following problems. You are not expected to finish all the problems. So take your time! :)

$$1. \text{ (a) } \lim_{n \rightarrow \infty} \frac{a^{n+1}}{(n+1)!} \quad (\text{where } a \text{ is some positive number}) = 0 \quad (\text{factorial beats exponential})$$

$$\text{ (b) } \lim_{n \rightarrow \infty} \frac{-3^{n+2}}{3^n + 10} = \lim_{n \rightarrow \infty} \frac{-3^a \cdot 3^n}{3^n + 10} = -9$$

Geometric Series ②

(a) What is the Taylor series of $f(x) = \frac{x}{3-5x^4}$?

(b) (from 2015 Final) For which values of x does the Taylor series for $\frac{x}{3-5x^4}$ converges?

(c) For which values of x does the series $\sum_{n=1}^{\infty} e^{-nx}$ converges?

$$(a) \frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$$

$$f(x) = \frac{x}{3-5x^4} = x \frac{1}{3[1-\frac{5}{3}x^4]}$$

$$= \frac{x}{3} \cdot \frac{1}{1-\frac{5}{3}x^4}$$

$$= \frac{x}{3} \sum_{n=0}^{\infty} \left(\frac{5}{3}x^4\right)^n$$

$$= \frac{x}{3} \sum_{n=0}^{\infty} \frac{5^n}{3^n} x^{4n} = \sum_{n=0}^{\infty} \frac{5^n}{3^{n+1}} x^{4n+1}$$

$$(b) u = \frac{5}{3}x^4$$

$$\left| \frac{5}{3}x^4 \right| < 1$$

$$\left| x^4 \right| < \frac{3}{5} \Rightarrow |x| < \sqrt[4]{\frac{3}{4}}$$

$$(c) \sum_{n=1}^{\infty} e^{-nx} = \sum_{n=1}^{\infty} \left(\frac{1}{e^x}\right)^n \quad \text{geometric series !!}$$

$$\left| \frac{1}{e^x} \right| < 1 \Rightarrow |e^x| < 1 \Rightarrow x > 0$$

3. (Convergence of Taylor Series) $f(x) = \sin x$. Find a bound for $|R_n f(x)|$ and use this to show that $T_n f(x) \rightarrow f(x)$. (in other words, $T_n f(x)$ converges to $f(x)$ as $n \rightarrow \infty$.)

$$|R_n f(x)| = \left| \frac{f^{(n+1)}(x)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}$$

(because $(n+1)$ -th derivative of $\sin x$ is either $\sin x, \cos x, -\sin x, -\cos x$, they are all bounded by 1 and 1,

Notice: $f(x) - T_n f(x) = R_n f(x)$

Since $|R_n f(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \rightarrow 0$ when $n \rightarrow \infty$ (refer to 1(a) on this worksheet).

So $R_n f(x) \rightarrow 0$ as $n \rightarrow \infty$

$\Rightarrow f(x) - T_n f(x) \rightarrow 0$ as $n \rightarrow \infty$. That is $T_n f(x) \rightarrow f(x)$ as $n \rightarrow \infty$. Done! :)

The following problems are about the Convergence test of Series.

4. Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

conv.

integral test.

(b) $\sum_{n=3}^{\infty} \frac{1}{10\sqrt{n}}$

div

integral test

(c) $\sum_{n=1}^{\infty} \frac{n!-8}{e^n+10}$

div.

Divergence test (also called term test) $\lim_{n \rightarrow \infty} \frac{n!-8}{e^n+10} = +\infty \neq 0$

(d) $\sum_{n=1}^{\infty} \left(\frac{n^3}{n!} \right)^n$

conv

root test

(e) $\sum_{n=10}^{\infty} \frac{1}{n^6+8n-1}$

conv.

limit comparison \oplus integral

(f) $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$

div

integral test

(g) $\sum_{n=1}^{\infty} \frac{1000^n}{n!}$

conv.

ratio $L = \lim_{n \rightarrow \infty} \frac{1000^n}{n!} = 0 < 1$

(h) $\sum_{n=1}^{\infty} \frac{n^3}{5n^3+n+1}$

div.

Divergence test $\lim_{n \rightarrow \infty} \frac{n^3}{5n^3+n+1} = \frac{1}{5} \neq 0$

(i) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

conv.

ratio $L = \lim_{n \rightarrow \infty} \frac{[(n+1)!]^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2}$
 $= \lim_{n \rightarrow \infty} \frac{(n+1)(n+1)}{(2n+1)(2n+2)} = \frac{1}{4} < 1$

(g)

Note that by a consequence of the Divergence test.

Since $\sum_{n=0}^{\infty} \frac{1000^n}{n!}$, then $\lim_{n \rightarrow \infty} \frac{1000^n}{n!}$. (this result agrees with the fact that factorial beats exponential.)

(i) Since $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ conv., we can conclude that $\lim_{n \rightarrow \infty} \frac{(n!)^2}{(2n)!} = 0$ (this solves P102 #10 in textbook.)